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Seepage flow in unconfined aquifers

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The paper concerns one- and two-dimensional models of steady seepage flow in unconfined aquifers and the relationship between them. The first part gives a new proof of Charnyi's result that one- and two-dimensional theory yield the same value for the flow rate in a horizontal aquifer or porous bed between vertical ends and shows the extent to which it can be generalized to non-uniform or anisotropic media. The second part solves the highly two-dimensional problem of flow from a line source (line of springs) in an otherwise impermeable, sloping stratum and compares the result with the predictions of a one-dimensional Dupuit–Pavlovsky approach. Confirmatory experiments using the Hele Shaw analogue of seepage flow are also reported.

1. Introduction

Seepage flow theory is generally based on Darcy's law

$$\mathbf{v} = -k \operatorname{grad}(y + p/\rho g), \tag{1}$$

in which $\mathbf{v}$ is the effective velocity, k is the permeability of the porous medium for the fluid in question, y is the height above some datum level, p is the fluid pressure (taken as the excess above atmospheric), ρ is the fluid density and g is the acceleration due to gravity. In steady flow it is normally acceptable to set $\operatorname{div}\mathbf{v} = 0$ in recognition of liquid incompressibility.

In confined aquifers or equivalent systems, where the boundaries of the region occupied by liquid are known *ab initio*, to solve the equations is relatively simple. In unconfined aquifers, however, the liquid has to be treated as having an unknown upper boundary or free surface within the porous medium, an idealization of the somewhat blurred water table that occurs between fully saturated and relatively dry ground under the agency of surface tension and other effects. This free surface makes the problem much more intractable and progress is often best made by reducing the number of dimensions from three to two or even to one.

This paper is concerned with steady seepage flows which are truly two-dimensional, all conditions being independent of one horizontal space co-ordinate (z), and with the question as to whether they can be adequately described by a one-dimensional or 'hydraulic' approach, in which quantities are treated as depending only on the horizontal co-ordinate x. This hydraulic approach is of some antiquity, going back to Dupuit (1863). It is nevertheless still very useful, particularly in relation to unsteady flows, where little progress has been made with exact two- or three-dimensional theory.

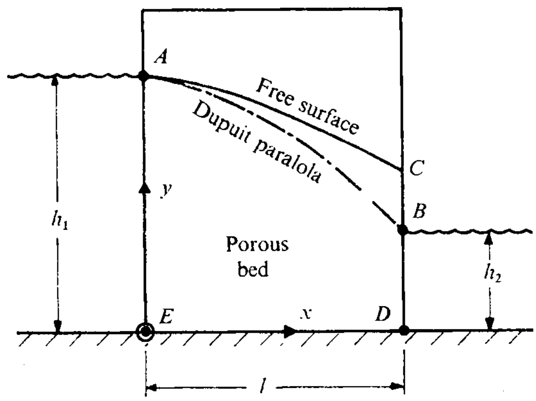

FIGURE 1. Seepage flow through a porous bed with vertical sides and a horizontal base.

The paper considers two different two-dimensional problems, and explores the applicability of one-dimensional theory. The first is a problem which Dupuit treated, concerning the flow through a porous bed on a horizontal impermeable base between two vertical sides adjoining pools of free liquid of different depth. As Charnyi (1951) showed, two-dimensional theory gives the same flow rate as Dupuit's theory, even though his model is very bad in detail. We give a new, briefer proof of this result and explore the limited scope for generalization to situations with non-uniform permeability, etc.

The second problem concerns the flow in an unconfined aquifer above an inclined impermeable layer in which there is a horizontal line source (e.g. a line of springs). This too is a highly two-dimensional flow. A simple one-dimensional model is set up and compared with the exact two-dimensional solution found by the inverse hodograph method and conformal transformation. A further comparison is provided by experiments performed on this configuration with the aid of the Hele Shaw analogy.

2. The Dupuit–Charnyi problem

The problem of steady flow in a vertical-sided porous bed on a horizontal impermeable base of length l between two pools of different depths h_1 and h_2, as shown in figure 1, was first solved, on a one-dimensional, 'hydraulic' basis, by Dupuit (1863), who took the free surface to be a curve (the Dupuit parabola) stretching from A to B. Dupuit's formula for the volumetric flow rate Q per unit thickness was

$$Q = k(h_1^2 - h_2^2)/2l. \tag{2}$$

Muskat (1937, pp. 316, 378) discussed the question of whether this result, based on the crude hydraulic model, could be correct despite the fact that the free surface of the liquid does not in practice encounter the downstream side at the level B of the downstream pool but emerges at a higher point C. BC is a seepage face from which liquid emerges and on which the pressure is atmospheric.

Charnyi (1951) showed that (2) agreed with two-dimensional theory (see for instance Polubarinova-Kochina 1962, chap. 7). Other authors, e.g. Hantush (1962), have also presented proofs. It is possible however to prove the result in a rather briefer manner which also clearly reveals the extent to which this simple result concerning a complicated two-dimensional problem can be generalized to allow for limited non-uniformity of the aquifer.

The horizontal component of the basic flow equation (1) is

$$v_x = -\frac{k}{\rho g}\frac{\partial p}{\partial x}.$$

We can integrate this equation over the domain $ACDE$:

$$\int dx \int v_x dy = -\frac{k}{\rho g}\int dy \int \frac{\partial p}{\partial x}dx, \tag{3}$$

in which a different order of integration has been used on the two sides. But $\int v_x dy = Q$ at each value of x and $\int (\partial p/\partial x)\,dx = p_2 - p_1$ at each value of y, where p_1 = pressure on AE and p_2 = pressure on $ACBD$ (zero, except on BD). Thus (3) becomes

$$Ql = \frac{k}{\rho g}\left\{\int_0^{h_1} p_1 dy - \int_0^{h_2} p_2 dy\right\} = \frac{k}{\rho g}\{\tfrac{1}{2}\rho g h_1^2 - \tfrac{1}{2}\rho g h_2^2\},$$

since p_1 and p_2 are given by hydrostatics in the form $p = \rho g(h-y)$. The Dupuit result (2) follows immediately.

This proof and result can be generalized to a limited extent by allowing the permeability to be anisotropic (but with principal directions vertical and horizontal) with the horizontal permeability k_x non-uniform but given by a function of the product form $k_x = X(x)\,Y(y)$. This of course includes the slightly more likely cases where k_x is a function of just *one* of x and y, owing to horizontal or vertical stratification. Such cases were treated by Irmay (1967) by longer methods. We can then write

$$\frac{v_x}{X} = -\frac{1}{\rho g}\,Y\frac{\partial p}{\partial x}$$

and integrate over the domain $ACDE$ to get

$$\int \frac{dx}{X}\int v_x dy = -\frac{1}{\rho g}\int Y\,dy \int \frac{\partial p}{\partial x}dx,$$

which gives

$$Q = \left\{\int_0^{h_1}(h_1 - y)\,Y\,dy - \int_0^{h_2}(h_2-y)\,Y\,dy\right\}\bigg/\left\{\int_0^l \frac{dx}{X}\right\},$$

which can be evaluated, given $X(x)$ and $Y(y)$.

This approach also clearly reveals why further exact generalization is not possible. If the principal directions of the permeability, if anisotropic, are inclined, v_x depends also on $\partial p/\partial y$ and the integral on the right-hand side involves knowledge of p along the base DE, for which no simple expression is available. The same consideration precludes generalization to cases where the base of the bed is inclined, as the relatively unsuccessful attempts at generalization of

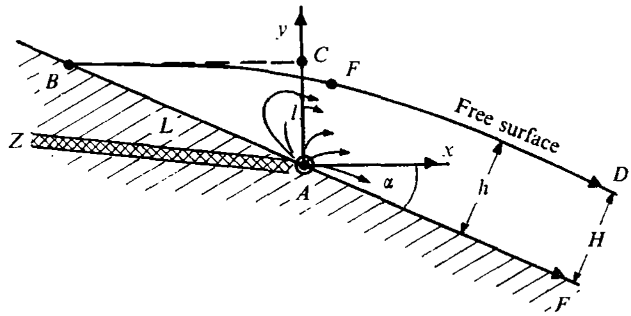

FIGURE 2. Seepage flow from a line source in the inclined base of an unconfined aquifer. The z plane.

Irmay (1967) reveal. Equally it is obvious that the upstream and downstream sides of the bed must be straight and vertical, for otherwise $\int v_x dy$ would not equal Q for those vertical intercepts through the bed which intersected a sloping side, and the method would fail.

The anisotropic case includes the case $k_x \ll k_y$ (where k_y is the vertical permeability), for which the Dupuit model is exact and the free surface is the Dupuit parabola. The vertical pressure distribution is hydrostatic, which is essentially the Dupuit assumption. One model of this situation is a series of vertical porous sheets with pools between.

Also included is the case $k_x \gg k_y$, where all streamlines are horizontal and C rises to the level of A. The bed behaves like a stack of horizontal capillary tubes. The flow along the uppermost streamlines will be very slight because of the small fall in head. This case reveals in its most extreme form the main reason why the Dupuit formula can work so well. The Dupuit model is self-compensating; it *underestimates* the depth available for flow at each vertical cross-section, but *overestimates* the horizontal velocities because it overestimates the fall in head along the streamlines.

3. Flow from a line of springs on a slope

The success of the Dupuit-type hydraulic model in predicting behaviour in what is essentially a two-dimensional problem in the previous section prompts the question as to whether such models can adequately describe other flows that are essentially two-dimensional. Figure 2 shows such a problem, which is of some practical interest, e.g. in establishing the extent of water percolation into soil overlying sloping rock.

Point A represents a horizontal line of buried springs in an otherwise impermeable, inclined, plane stratum, such as might occur where a thin confined aquifer ZA ends. The liquid emerges into an unconfined, isotropic and uniform aquifer and ultimately flows uniformly downhill at a depth H and effective velocity $k \sin \alpha$, α being the inclination of the stratum. Inevitably the liquid backs up the slope through some distance AB, equal to L, say, above the line of springs. The main problem is to determine L and H and the relation between them and the inclination α. The flow near the springs is highly two-dimensional; neverthe-

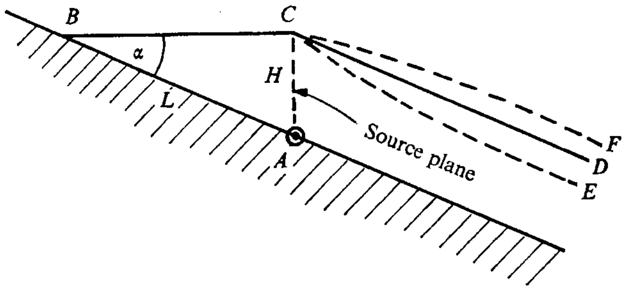

FIGURE 3. The one-dimensional, hydraulic model.

less we shall explore whether it can be meaningfully modelled in the 'hydraulic' manner, before solving the problem exactly, both analytically and experimentally by means of the Hele Shaw analogue.

The extension of Dupuit theory to inclined aquifers was done in 1931 by Pavlovsky (1956) on the assumption that the inclination of the base stratum and of all streamlines was small. The velocity was treated as virtually uniform over each vertical section. Pavlovsky's solutions reveal that the depth of liquid can only asymptote to a steady value in the *uphill* direction, a key point which is taken further below. See, for instance, Harr (1962, p. 44).

The line of springs in figure 2, in order to be modelled in a manner consistent with the Dupuit–Pavlovsky approach, has to be replaced by an imaginary vertical plane AC of sources distributed uniformly over the full depth of liquid in the aquifer as in figure 3. All the flow must go to the right as there is no outlet to the left, where the fluid must be at rest, having therefore a horizontal free surface CB. As regards the downstream behaviour, a free-surface curve such as CE or CF which asymptotes to a steady depth is precluded by the property of Pavlovsky solutions already referred to. The only possibility is the *immediate* onset of the steady depth H as shown by the free-surface line CD. (Whether H is measured vertically or normal to the inclined base is immaterial in this approximation.) The free surface BCD is therefore the Dupuit–Pavlovsky-style solution of this problem. The velocity of the fluid to the right of A is $k \sin \alpha$, or $k\alpha$ since α is small, and so $Q = kH\alpha$. The ratio of the back-up distance L to the steady depth H is given by

$$L/H = \operatorname{cosec} \alpha \doteqdot 1/\alpha,$$

since α is small. We see that the hydraulic model yields a very simple, if apparently crude solution to the problem. The next step is to solve the problem exactly and make a comparison.

4. Two-dimensional theory

We take the origin at the source A in figure 2, which is the z plane, where $z = x+iy$. The problem will be solved by complex-variable methods using a velocity potential $\phi = -k(y+p/\rho g)$, such that $\mathbf{v} = \operatorname{grad} \phi$, and a conjugate stream function ψ, the flow being incompressible. The complex potential w is $\phi + i\psi$ and is an analytic function of z, and vice versa.

The boundary conditions are that $\psi = 0$, say, along AB and the free surface

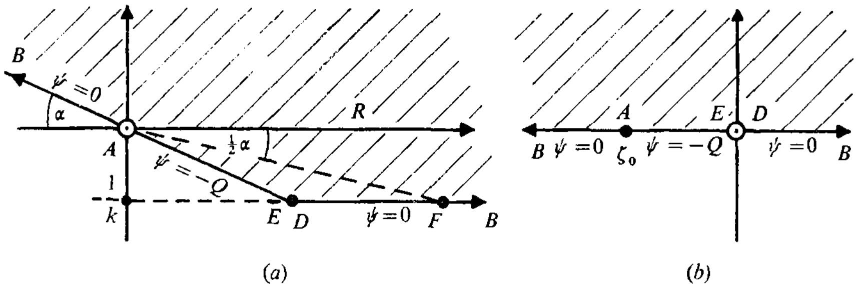

FIGURE 4. (a) The inverse hodograph (dz/dw) plane and (b) a transformation of it, the ζ plane.

BFD, and $\psi = -Q$ along AE. (F is a point that will be discussed later.) ϕ ranges from $-\infty$ at A (as is usual for sources) to $+\infty$ at DE, far downstream, and will be taken as zero at B. As the free surface is not determined *ab initio*, a further condition there is required, namely $p = 0$ (p being the gauge pressure) or $\phi + ky = 0$. Hence, along the free surface, with ψ constant, we have

$$\frac{dz}{dw} = \frac{\partial z}{\partial \phi} = \frac{\partial x}{\partial \phi} + i\frac{\partial y}{\partial \phi} = \frac{\partial x}{\partial \phi} - \frac{i}{k}.$$

Thus, on the free surface,

$$\operatorname{Im}(dz/dw) = -1/k, \quad \text{a constant.} \tag{4}$$

This fact provides the key to the success of the inverse hodograph method for solving unconfined aquifer problems, because the free surface, unknown in the z plane, becomes a known line in the dz/dw plane. Then dz/dw, which is a function of w, can be related to w by conformal transformation. Integration then gives z in terms of w and the problem is solved. The relation between dz/dw and the velocity components (v_x, v_y) is

$$\frac{dz}{dw} = \frac{v_x + iv_y}{v_x^2 + v_y^2} \quad \left(\text{for} \quad \frac{dw}{dz} = v_x - iv_y\right). \tag{5}$$

The position vector for dz/dw in the dz/dw plane (figure 4*a*) is therefore parallel to the corresponding velocity vector.

We shall use the same letters to denote corresponding points and boundaries on the physical (z) plane, on the dz/dw plane and on transformations of that plane. Then BFD is part of the line $\operatorname{Im}(dz/dw) = -1/k$. Along BAE, which is a streamline,

$$\frac{\operatorname{Re}(dz/dw)}{\operatorname{Im}(dz/dw)} = \frac{v_x}{v_y} = -\cot\alpha,$$

and so BAE is also a line of slope α through the origin in the dz/dw plane. At the source A, the velocity tends to infinity and $dz/dw \to 0$, i.e. A is the origin in the dz/dw plane. B is a stagnation point where $|dz/dw| \to \infty$ and on the dz/dw plane is a range of points at infinity, for the velocity direction is indeterminate there. These facts suffice to define the region of interest on the dz/dw plane, for the velocities out of the source will be generally upwards and/or rightwards and then (5) shows that dz/dw will lie in the region shown shaded in figure 4 (a). BFD is that part of the line defined by (4) that lies to the right of BAE. On this inverse hodograph plane the points D and E coincide, for the velocities at all points far downstream are the same.

w is determined as a function of dz/dw by the fact that $\psi = 0$ along AB and BD and $\psi = -Q$ along AE, while $\phi \to -\infty$ at the source A and $\phi \to +\infty$ at the 'sink' D/E. Progress is best made by a conformal transformation of the relevant part of the dz/dw plane into the upper half of the ζ plane shown in figure 4 (*b*). The required transformation, which expands the angle ADB from $\pi - \alpha$ to π and puts the ζ origin at D/E is

$$\zeta^n = \frac{dz}{dw} - \frac{e^{-i\alpha}}{k \sin \alpha}, \quad \text{where} \quad n = \frac{\pi - \alpha}{\pi},$$

D/E being the point where $dz/dw = e^{-i\alpha}/(k \sin \alpha)$. The source A is now at $\zeta = \zeta_0$, where

$$-\zeta_0 = (k \sin \alpha)^{-1/n}.$$

Since B is now the complete upper semicircle at infinity in figure 4 (*b*) and $\psi = 0$ there, the solution for w in the upper half-plane is merely that due to a source at A and a sink at D/E, namely

$$w = \frac{Q}{\pi} \{\log(\zeta - \zeta_0) - \log \zeta\} = \frac{Q}{\pi} \log\left(1 - \frac{\zeta_0}{\zeta}\right).$$

Hence
$$\zeta = -\zeta_0/(e^{\pi w/Q} - 1)$$

and
$$\frac{dz}{dw} = \frac{1}{k \sin \alpha}\left\{\frac{1}{(e^{\pi w/Q} - 1)^n} + e^{-i\alpha}\right\}. \tag{6}$$

As we have chosen to make $\phi = 0$ at B, the point where $z = -Le^{-i\alpha}$, then $w = 0$ there also and (6) integrates to

$$(k \sin \alpha) z = \int_0^w \frac{dw}{(e^{\pi w/Q} - 1)^n} + e^{-i\alpha}(w - kL \sin \alpha). \tag{7}$$

In order to find L, the back-up distance, we note that $w = \phi$ along AB and is negative and real. Thus $e^{\pi w/Q} < 1$ and on AB

$$\frac{1}{(e^{\pi w/Q} - 1)^n} = \left(\frac{e^{i\pi}}{1 - e^{\pi\phi/Q}}\right)^n = \frac{e^{i(\pi - \alpha)}}{(1 - e^{\pi\phi/Q})^n} = -\frac{e^{-i\alpha}}{(1 - e^{\pi\phi/Q})^n},$$

and
$$(k \sin \alpha) z = -e^{-i\alpha}\left[\int_0^\phi \left(\frac{1}{(1 - e^{\pi\phi/Q})^n} - 1\right) d\phi + kL \sin \alpha\right].$$

Note that this equation correctly locates z in the second quadrant with argument $\pi - \alpha$, for the term in square brackets is real and positive. We may find L by setting $z = 0$ and letting $\phi \to -\infty$ at A, with the result that

$$kL \sin \alpha = \frac{Q}{\pi} \int_{-\infty}^0 \left\{\frac{1}{(1 - e^{\pi\phi/Q})^n} - 1\right\} d(\pi\phi/Q) = -\frac{Q}{\pi}\left\{\psi\left(\frac{\alpha}{\pi}\right) + C\right\},$$

in which ψ is Euler's 'psi' function and C is Euler's constant. Values are tabulated in Jahnke, Emde & Lösch (1960, p. 15). $\psi + C$ is easily evaluated for $\alpha = \frac{1}{2}\pi, \frac{1}{3}\pi$, etc.

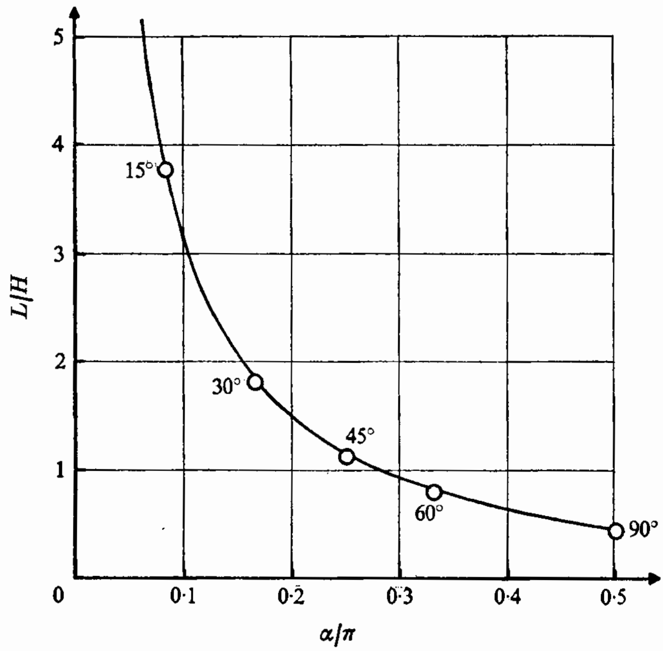

FIGURE 5. Back-up distance ratio L/H as a function of inclination. The curve is theoretical, the points experimental.

Along the free surface, $w = \phi$, rising from 0 at B to $+\infty$ at D. Then the integral in (7) is real and the imaginary part of (7) confirms that $\phi + ky =$ constant. It can also be easily deduced that the thickness of the liquid layer normal to the sloping base (h in figure 2) is simply expressible as the integral

$$h = \frac{Q}{\pi k}\int_0^{\phi} \frac{d(\pi\phi/Q)}{(e^{\pi\phi/Q}-1)^n}.$$

As $\phi \to \infty$, the integral takes the simple value $\pi/\sin\alpha$, which is consistent with the uniform terminal velocity $k\sin\alpha$ down the slope.

The most obvious simple parameter that characterizes the two-dimensional flow in the unconfined aquifer is the ratio of the back-up distance L to the ultimate depth H of the flow, measured normal to the bed, which equals $Q/k\sin\alpha$. The result is

$$L/H = -\{\psi(\alpha/\pi) + C\}/\pi. \tag{8}$$

This quantity is plotted as a function of α/π in figure 5.

The case $\alpha = \frac{1}{2}\pi$ may also be regarded as the double-sided flow from a line source such as a leaking duct buried in a virtually unlimited porous bed, i.e. with the stratum BAE, now vertical, replaced by a plane of symmetry. This is a problem which has been solved before (see, for instance, Polubarinova-Kochina 1962, p. 181). In this case, at the free surface,

$$x = (2Q/k\pi)\tan^{-1}(e^{\pi\phi/Q}-1)^{\frac{1}{2}}, \quad \phi + ky = kL$$

and

$$L - y = (2Q/k\pi)\log\sec(k\pi x/2Q).$$

This solution can obviously also be adapted to solving the problem of two-dimensional flow into an inclined *confined* aquifer of small uniform thickness from a point source, the x, y plane then being that of the aquifer. k must be replaced by $k \sin \beta$, where β is the slope of the aquifer.

The three-dimensional problem of flow into an inclined *unconfined* aquifer from a *point* source has not been solved. Approximate solutions for the axisymmetric case of this type, where $\alpha = 90^\circ$ and the stratum, now vertical, may be replaced by a plane of symmetry, have been discussed by Polubarinova-Kochina (1962, pp. 395 *et seq.*).

Although for general values of α the values of the integral giving the shape of the free surface are not readily available, the particular point F in figure 2, where $\pi\phi/Q = \log 2$, can be more fully documented in terms of the standard function $\beta(\alpha/\pi)$, where β is defined by

$$\beta(p) = \tfrac{1}{2}\{\psi(\tfrac{1}{2}(p+1)) - \psi(\tfrac{1}{2}p)\}.$$

The co-ordinates of F are such that

$$\left.\begin{aligned} y &= L\sin\alpha - (Q/k\pi)\log 2 \\ \text{and} \qquad (k\pi/Q)\,x\sin\alpha &= \beta(\alpha/\pi) - \cos\alpha[-\{\psi(\alpha/\pi) + C\} - \log 2].\end{aligned}\right\} \tag{9}$$

It is shown later that $dy/dx = -\tan\tfrac{1}{2}\alpha$ at F, i.e. F is the point where the free surface has tilted half-way towards its final inclination α. This locates it on figure 4 (a) as the point on BD where $\angle RAF$ is $\tfrac{1}{2}\alpha$. We refer again to F later.

5. A comparison with experiment

The well-known Hele Shaw analogy provides the most convenient technique for performing experiments on two-dimensional seepage flow in unconfined aquifers. A rectangular glass-sided Hele Shaw tank with a gap of 0·5 mm and other dimensions 110×700 mm was employed in the experiments reported here. The source A was a transverse inlet pipe of diameter 5 mm, centred on the lower edge of one of the glass plates, at a distance of 230 mm from the upstream corner. The inclination α of the tank could be varied from 0 to 90°, measured with a sensitive clinometer. It was found to be a relatively simple matter to achieve steady flow, with the downstream flow thickness asymptoting to a virtually steady value H after a downhill distance of about $3H$ from the source. This rose to about $6H$ as α approached 90°. Only at the highest flow rates was there any doubt whether the asymptotic state had been reached within the apparatus. As low values of α were approached, the change in slope of the free surface was observed to become increasingly abrupt, as suggested by the Dupuit–Pavlovsky model shown in figure 3. The liquid used was methylated spirits, whose surface-tension behaviour is much less fickle than that of water. Unlike some unconfined aquifer simulations with the Hele Shaw analogue, this one is such that surface tension has little deleterious effect although at values of α greater than 60° the location of the point B became somewhat indeterminate because surface tension caused a small rise just as the free surface finally approached the 'stratum' BAE. This final rise was ignored in estimating the position of B.

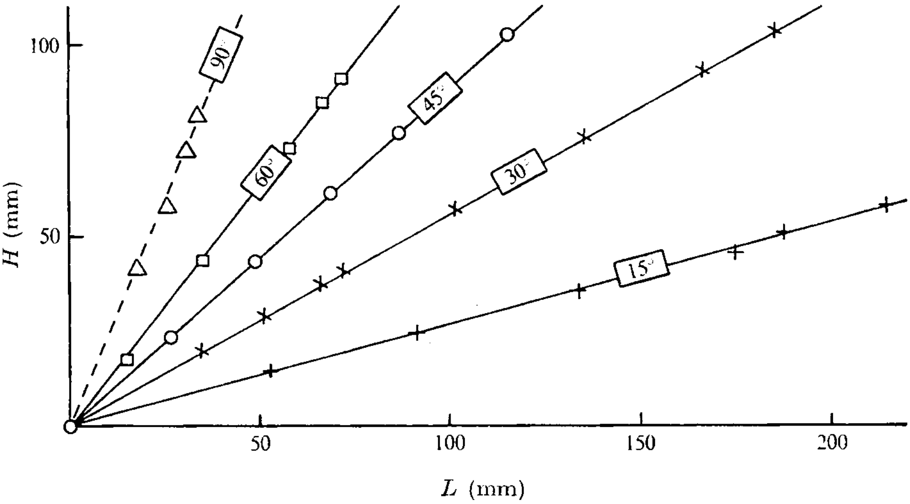

FIGURE 6. Ultimate depth H plotted against back-up distance for various inclinations, measured in the Hele Shaw experiments.

The quantities L and H were measured to an accuracy of 0·5 mm by visual observation through the glass. The results are presented in figure 6 as graphs of H against L for various inclinations α. The graphs show H to be proportional to L within the accuracy expected, except for the case $\alpha = 90°$. It became evident that to do fully satisfactory experiments on this case it would be necessary to simulate the double-sided flow mentioned in § 4, in order to eliminate the surface-tension rise at B. At the highest flow rates, with $\alpha = 90°$, the asymptotic value of H was not reached within the apparatus. Even at 90° the free surface was completely stable, but when α exceeded 90°, a beautiful nonlinear wavelike convective instability, with a strongly preferred wavelength, set in (see figure 7, plate 1). This does not represent a phenomenon that would occur with groundwater, however.

The slopes of the best straight lines in figure 6 yield experimental values for L/H, which are plotted on figure 5. The agreement with theory is very satisfactory.

The Reynolds number of the flow based on gap width and the properties of methylated spirits is

$$\rho^2 g d^3 \sin\alpha / 12\mu^2 = 61 \sin\alpha.$$

This is sufficiently low to ensure the laminar flow on which the Hele Shaw analogy depends. The value of viscosity for the methylated spirits used was in fact deduced from measurements of flow rates in the apparatus ($\mu = 1{\cdot}06 \times 10^{-3}$ in S.I. units).

6. A comparison with Dupuit–Pavlovsky theory

The question remains as to how well the crude 'hydraulic' model discussed in § 3 corresponds to the exact solution when α becomes small. We shall make use of the expansions

$$-(\psi(p)+C) = p^{-1}-\tfrac{1}{6}\pi^2 p+O(p^2) \tag{10}$$

and

$$\beta(p) = p^{-1}-\log 2+\tfrac{1}{12}\pi^2 p+O(p^2). \tag{11}$$

Combining (8) and (10) gives $L/H = \alpha^{-1}(1-\frac{1}{6}\alpha^2\ldots)$ and the result of the hydraulic approximation $L/H = 1/\alpha$ is seen to be correct to second order, for small α. First-order errors might well have occurred.

A more detailed check is provided by a consideration of the point F, referred to in § 4. We shall examine its position relative to the point C, and take as our reference length scale the distance AC, which equals $L\sin\alpha$, or l, say. The relation between Q, l and α is

$$lk/Q = \alpha^{-1}\{1-\tfrac{1}{6}\alpha^2+O(\alpha^3)\}.$$

Note that Q is $O(\alpha)$ for given l. The position of F relative to C is given from (9)–(11) by the relations

$$\frac{l-y}{l} = \frac{Q}{k\pi l}\log 2 = \frac{\alpha}{\pi}(\log 2)(1+O(\alpha^2))$$

and

$$k\pi x\sin\alpha/Q = \tfrac{3}{4}\pi\alpha+O(\alpha^2), \quad x/l = \tfrac{3}{4}\alpha(1+O(\alpha)).$$

As $\alpha\to 0$, with l constant, F tends in a regular manner to C, its equivalent in the Dupuit–Pavlovsky model.

The slope of the free surface deduced from (7) with $w = \phi$ (positive) is

$$-\frac{dy}{dx} = \frac{1}{k}\frac{d\phi}{dx} = \frac{\sin\alpha}{\cos\alpha+(e^{\pi\phi/Q}-1)^{-n}} \quad \left(\text{where } n = 1-\frac{\alpha}{\pi}\right), \tag{12}$$

which changes from 0 to $\tan\alpha$ as $\pi\phi/Q$ rises from 0 at B to ∞ at D. It equals $\tan\frac{1}{2}\alpha$ when $\pi\phi/Q = \log 2$, at F, as remarked earlier. For small $\pi\phi/Q$ we have the approximation

$$k^{-1}d\phi/dx = \sin\alpha(\pi\phi/Q)^{1-\alpha/\pi}.$$

Integrating from B, where $\phi = 0$ and $x = x_B$, say, gives

$$(\pi\phi/Q)^{\alpha/\pi} = (k\alpha/Q)\sin\alpha\,(x-x_B) \doteqdot (x-x_B)/L,$$

if α is small and $Q \doteqdot \alpha^2 Lk$. This relation implies that, for α small, ϕ hardly changes as x increases from x_B until $(x-x_B)/L$ gets close to unity in the vicinity of C and F. Then $\pi\phi/Q$ rises rapidly towards unity and the preceding approximation fails.

If ϕ is constant, so is y, and the flat upstream surface predicted by the hydraulic model is confirmed. As ϕ is the velocity potential, ϕ constant at constant y implies zero horizontal velocity.

As $\pi\phi/Q$ rises past unity, the strong exponential term in (12) causes dy/dx to undergo most of its change as $\pi\phi/Q$ rises from $\frac{1}{3}$ to 3 say, as the point F (where

$\pi\phi/Q = 0{\cdot}693$) is passed. In this vicinity $d\phi/dx$ is of order $k\alpha$ and so most of the change in dy/dx occurs over a range of x values of order $Q/k\alpha$, i.e. of order l. This confirms what was also apparent from the experiments, namely, that for small α, the swift transition from one straight part of the free surface to the other takes place over a horizontal distance of the same order as the depth of the layer. This is of course the zone where the flow is truly two-dimensional. Outside it the fluid is either virtually at rest or moving uniformly down the slope.

To sum up this section, the one-dimensional hydraulic model is found to give acceptable predictions of the back-up distance and the shape of the free surface despite the fact that the flow is highly two-dimensional near the source.

7. Final remarks

The second problem is probably typical of truly two-dimensional problems that can be well modelled by one-dimensional theory provided that the inclinations of the base stratum and of the free surface are small. The first problem is probably unique in that there the one- and two-dimensional theories predict exactly the same flow rate (as distinct from agreeing asymptotically as the slope tends to zero) for all inclinations of the free surface. It also offers unique scope for generalization to non-uniform or anistropic media.

REFERENCES

CHARNYI, I. A. 1951 *Doklady*, **79** (6).

DUPUIT, J. 1863 *Etudes Théoriques et Pratiques sur le Mouvement des Eaux*. Paris: Dunod.

HANTUSH, M. 1962 *J. Geophys. Res.* **62**, 2419.

HARR, M. E. 1962 *Groundwater and Seepage*. McGraw-Hill.

IRMAY, S. 1967 *Water Resources Res.* **3**, 599.

JAHNKE, E., EMDE, F. & LÖSCH, F. 1960 *Tables of Higher Functions*, 6th edn. McGraw-Hill.

MUSKAT, M. 1937 *The Flow of Homogeneous Fluids through Porous Media*. Ann Arbor, Michigan: J. W. Edwards Inc.

PAVLOVSKY, N. N. 1956 *Collected Works*. Leningrad: Akad. Nauk.

POLUBARINOVA-KOCHINA, P. YA. 1962 *Theory of Groundwater Movement* (trans. J. M. Roger de Wiest). Princeton University Press.

FIGURE 7. Instability of the free surface in the Hele Shaw experiment for $\alpha > 90°$. The liquid is to the right and the source is beyond the top of the picture.